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REGULAR AND SINGULAR SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

ABSTRACT

of the PhD thesis for the degree of
doctor of Philosophy (PhD) in the specialty
«6D060100-Mathematics»

The relevance of the research topic. Differential operators of fractional order can be defined by different ways, and therefore, when solving boundary value problems and partial differential equations of fractional order, depending on the proposed operators they should have different approaches.

We should also note that, using operators of fractional order, one can deeper understand the known results of the theory of functions and boundary value problems, and get a new class of solutions that allows to cover a wide range of problems.

In 1827, the Scottish botanist R. Brown, observing the chaotic motion of pollen particles in a drop of water, discovered a new type of movement, later named after him, the Brownian motion. The main laws of Brownian motion are the Gaussian form of the diffusion packet and its spreading according to the law $t^{1/2}$ seemed as universal as Newton's laws. However, in 1926, L. Richardson found that under the turbulent medium conditions the diffusion packet spreads out according to the law $t^{3/2}$. Later this process was called super-diffusion. Another type of diffusion that differs from normal diffusion is sub-diffusion (slow diffusion). It was found in percolation processes, in the relaxation of solid materials, in polymer structures, other phenomena and systems. It turned out that the law of expansion, the form of the diffusion packet and the type of equation describing these processes, differ substantially from the normal case. This class of phenomena was called anomalous diffusion or strange kinetics.

In the 90s of the last century, it became clear that an important place in the theory of anomalous diffusion is occupied by equations with fractional derivatives. Therefore, the study of various problems for equations with fractional derivatives are relevant from the point of view of applied science.

In general, one of the main popular methods widely used in the study of the theory of partial differential equations is the principle of maximum or minimum.

This principle allows to get information about sought solutions without knowing about their specific forms.

Recently, due to the development of problems of fractional differential equations, special attention has been paid to the maximum principle.

One of the important directions in the theory of nonlinear problems is the global solvability and blow-up of solutions in finite time. The classical theory of global solvability of non-linear equations is essentially devoted to sufficient conditions ensuring the solvability of the corresponding initial and boundary value problems. This new phenomenon in nonlinear global analysis was discovered relatively recently. It began with the classical work of the Japanese mathematician H. Fujita and was called the blow-up of solutions.

In 1966, H. Fujita proved the existence of a global solution for a nonlinear heat equation and a critical index for the blow-up of the solution in finite time. In his honor, this indicator was called the "critical exponent of Fujita".

Up to now, various generalizations of Fujita's results have been studied in many papers. Critical exponents of the Fujita type for partial analogs of the diffusion equation and the integral nonlinear diffusion equation are also studied.

The relevance of the chosen research topic indicates the rapid development of research in this direction, as well as the fact that currently many works on differential and integro-differential equations of fractional order are presented on the authoritative international databases Web of Science, Scopus, MathSciNet, etc.

The aim of the PhD thesis is to study of regular and singular solutions of systems of linear and nonlinear fractional differential equations and equations.

To achieve the aim of the dissertation, the main objectives of the following research are considered:

- studying of the maximum and minimum principle for the diffusion equation, which involved differential operators of fractional order in the value of Riemann-Liouville, Caputo-Fabrizio and generalized Caputo-Fabrizio;
- obtaining an analogue of the principle Duhamel for the diffusion equation of linear fractional order;
- investigating of the existence of local solutions of equations and systems of equations of non-local and weightless nonlinear diffusion;
- diffusion equation of exponential nonlinear fractional order and investigation of local and global solvability of systems of equations;
- finding conditions for the nonexistence of a global solution for the diffusion equation and a system of equations of polynomial nonlinear fractional order, i.e. critical indicators of the Fujita type.

Object of the PhD thesis. The differential operators of fractional order in the value of Riemann-Liouville, Caputo-Fabrizio, and generalized Caputo-Fabrizio represent the diffusion equation given by the initial boundary conditions. Also consider integro-differential equations and systems of fractional order with exponential and polynomial nonlinearity.

The methods of scientific research. The researches on the theme assume the usage of the classical methods of the theory of partial differential equations.

In studying the properties of solutions to nonlinear integro-differential diffusion equations, we will use various methods of the theory of partial differential equations and nonlinear analysis. Here are some of the methods that are planned to use solutions to the problems.

To prove the existence of local solutions, the following are used:

- Method of reduction to integral equations and Duhamel's principle;
- Methods of contraction mappings;
- Methods of fractional calculus;
- Fixed point theorems.

The existence of global solutions will be proved using the methods:

- Method of fundamental solutions and method of Green's functions;
- Inequalities for fractional derivatives;
- Fixed point theorems;

To investigate the blow-up of solutions will be used:

- Test functions method;

Scientific novelty of the work. In this dissertation, questions of mathematical physics involving derivatives of linear and nonlinear fractional order are investigated, and also include classical linear problems, as well as a class of nonlinear problems with significant applications. The issues under consideration are mostly not studied or marked only for independent cases. Therefore, the research work summarizes the known results.

Theoretical and practical significance of the results. The research work is mainly theoretical and fundamental. Therefore, the scientific significance of the work is associated with the use of deep, modern results of the theory of differential operators and the creation of new proprietary research and analysis methods.

Publications. On the topic of the thesis 12 papers were published, including 4 publications in a high-ranking scientific journal, indexed in the Web of Science and Scopus, 3 publications in scientific journals included in the list recommended by the Committee on the Control of Education and Science of the MES RK for publication of the main scientific results of scientific activities, 2 publications on the rights of the manuscript, 1 publication in a domestic journal, 4 publications in materials of foreign international conferences**.**

The results on the topic of the thesis were published in the following papers:

Publication in the high-ranking scientific journals

1 M. Borikhanov, M. Kirane, B. T. Torebek. Maximum principle and its application for the nonlinear time-fractional diffusion equations with Cauchy-Dirichlet conditions //Applied Mathematics Letters, - 2018. - V. 81, - P. 14-20. Web of Science Impact factor=3,8(Q1), JIF Percentile=98, Scopus SJR=1,4(Q1), CiteScore=5,7, Scopus Percentile=94.

2 Borikhanov M., Torebek B. Local existence and global nonexistence results for an integro-differential diffusion system with nonlocal nonlinearities // Mathematical Methods in the Applied Sciences - 2020. - P. 1796-1811. doi.org/10.1002/mma.6878. Web of Science Impact factor=1,6(Q2), JIF Percentile=74, Scopus SJR=0,6(Q1), CiteScore=2,8, Scopus Percentile=87.

CCES

1 M. Borikhanov, B. T. Torebek. Maximum principle and its application for the sub-diffusion equations with Caputo-Fabrizio fractional derivative //Математический журнал. - 2018. - Т. 18, № 1. - С. 43-52.

2 Borikhanov M., Torebek B. Critical exponents of Fujita type for certain time-fractional diffusion equations // International Journal of Mathematics and Physics. - 2018. - V. 9, No. 2. - P. 43-49.

3 M. B. Borikhanov. Mild solution to integro-differential diffusion system with nonlocal source. //Kazakh Mathematical Journal. - 2020. - Vol. 20, No. 1. - P. 18-26.

On the rights of the manuscript

1 Borikhanov M., Torebek B. Local and blowing-up solutions for an integro-differential diffusion equation and system. arXiv:1910.06989.

2 M. Borikhanov, M. Kirane, B. T. Torebek. Globally unsolvability of integro-differential diffusion equations with exponential nonlinearity. Researchgate. DOI: 10.13140/RG.2.2.18342.14409

Kazakh local journal

1 Borikhanov M.B. Critical exponents of Fujita type for system of time-fractional diffusion equations. // Proceedings Of the IKTU named after H. A.Yasavi (mathematics, physics, computer science series) - 2018. - Т. 3, №6. - С. 23-35.

Publications in materials of international conferences

1 M. B. Borikhanov. Maximum principle and its application for the nonlinear time-fractional Stokes's first problem // Traditional International April scientific conference in honor of the Science Day. Almaty, April 10, - 2018. - P. 27.

2 Borikhanov M.B. Maximum principle and its application for the nonlinear time-fractional diffusion equations // Современные методы теории краевых задач: материалы международной конференции «Понтрягинские чтения - XXIX», посвященной 90 - летию Владимира Александровича Ильина (2-6 мая 2018 г.) - 2018. Москва. - С. 251-252.

3 M. Borikhanov. Duhamel principle for the time-fractional diffusion equation in unbounded domain // Fourth International Conference on Analysis and Applied Mathematics (ICAAM 2018). - 2018. 6-9 September. Near East University, Lefkosa (Nicosia), Mersin 10, Turkey. - P. 129.

4 Borikhanov M. Local existence and global non-existence for the integro-differential diffusion equation // International Conference “Actual Problems of Analysis, Differential Equations and Algebra”. - 2019. Nur-Sultan - P. 86.

The structure and scope of the thesis. The PhD thesis includes a title page, content, introduction, four chapters, conclusion and list of references, consisting of 55 titles. The total volume of the thesis is 102 pages.

The main content of the thesis.

In Chapter 1, we considered an initial-boundary fractional diffusion equation with $D_{0|t}^{1-\alpha}$ Riemann-Liouville fractional derivative

$$\begin{cases} u_t(x,t) = \Delta_x D_{0|t}^{1-\alpha} u(x,t) + F(x,t), \ (x,t) \in (0,a) \times (0,T] = \Omega, \\ u(x,0) = \varphi(x), \ x \in [0,a], \\ u(0,t) = \lambda(t), \ u(a,t) = \mu(t), \ 0 \le t \le T, \end{cases}$$

where $0 < \alpha < 1$, $F(x,t)$, $\varphi(x)$ are continuous functions.

Also, we study an initial-boundary fractional diffusion equation with ${}_{CF}\mathbf{D}_{0|t}^{1-\alpha}$ in the sense Caputo-Fabrizio fractional derivative

$$\begin{cases} u_t(x,t) = \dfrac{\partial^2}{\partial x^2} {}_{CF}\mathbf{D}_{0|t}^{1-\alpha} u(x,t) + F(x,t), \ (x,t) \in (0,a) \times (0,T] = \Omega, \\ u(x,0) = \varphi(x), \ x \in [0,a], \\ u(0,t) = u(a,t) = 0, \ 0 \le t \le T, \end{cases}$$

where $0 < \alpha < 1$, $F(x,t)$, $\varphi(x)$ are continuous functions $\varphi(0) = \varphi(a) = 0$.

In addition, we studied an initial-boundary fractional diffusion equation with ${}_{CF}^{*}\mathbf{D}_{0|t}^{1-\alpha}$ generalized Caputo-Fabrizio fractional derivative

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t)=\dfrac{\partial^2}{\partial x^2}\,{}_{CF}^{*}\mathbf{D}_{0|t}^{1-\alpha}u(x,t)+F(x,t),\ (x,t)\in(0,a)\times(0,T]=\Omega, \\ u(x,0)=\varphi(x),\ x\in[0,a], \\ u(0,t)=\lambda(t),\ u(a,t)=\mu(t),\ 0\le t\le T, \end{cases}$$

where $0<\alpha<1,\ F(x,t),\varphi(x),\lambda(t),\mu(t)$ are continuous functions and $\lambda(t),\mu(t)$ are nondecreasing functions. For these equations maximum and minimum principle have been obtained.

As an application of the obtained results, we prove the uniqueness of solutions to linear and nonlinear diffusion equations, as well as the continuous dependence of solutions on the original set ones.

In Chapter 2, an analog of the Duhamel principle is obtained for the fractional order diffusion equation in time:

$$\begin{cases} u_t(x,t)-\Delta_x D_{0|t}^{1-\alpha}u(x,t)=f(x,t),\ \ 0<\alpha<1,\ \ x\in R^N,\ \ t>0, \\ u(x,0)=u_0(x),\ \ x\in R^N, \end{cases}$$

where $u_0(x)$ and $f(x,t)$ are given functions.

The existence of local solutions of nonlocal equations and nonlinear systems of diffusion equations with weight are also investigated.

Using an analog of the Duhamel principle, taking into account the properties of the green function of the fractional order diffusion equation in time and the Banach fixed point theorem, we prove that the problem has a unique local integral solution.

In Chapter 3, for a polynomial nonlinear diffusion equation and a system of fractional order equations, the conditions for the blow-up (explosion) of the global solution are determined, i.e., the values of critical exponents of the Fujita type are determined.

By the method of test functions, as well as by calculating fractional order and inequalities for fractional order derivatives, the Fujita type critical exponents were found.

The nonexistence of a global solution of equations and systems of fractional-order diffusion equations with exponential nonlinearity is investigated and proved by the method of test functions.